

Today X stands for an integral (reduced, irreducible) noetherian scheme.

Def. The group of Weil divisors is the additive free abelian group spanned by the irreducible closed subsets of codim 1. It's denoted $\text{Weil}(X)$.

• A member of the group of Weil divisors is called a Weil divisor. So a Weil divisor is a formal sum $\sum_{i=1}^t n_i Z_i$, where Z_i irr and $\text{codim}_X(Z_i) = 1$, $n_i \in \mathbb{Z}$

$$\sum_{i=1}^t n_i Z_i + \sum_{i=1}^t n'_i Z_i = \sum_{i=1}^t (n_i + n'_i) Z_i$$

Ex: In $\mathbb{P}^1_{\mathbb{C}}$, codim 1 pts are closed pts so a Weil divisor looks like $\sum n_i ([x_i : y_i])$, such as $[1:0] - [0:1]$.

Ex. In $\mathbb{A}^2_{\mathbb{C}} = \text{Spec}(\mathbb{C}[x,y])$ $V(x) - 3V(y)$ is a Weil div.



Ex. Take the rational function $X/Y \in \mathbb{C}(X,Y)$ on $\mathbb{A}^2_{\mathbb{C}}$. $V(x) - 3V(y)$ reads the zeros and poles of this rational function with order.

For today unless otherwise mentioned we shall assume X is regular in codimension 1. This means for every $\xi \in X$ such that $\dim \mathcal{O}_{X,\xi} = 1$, $\mathcal{O}_{X,\xi}$ is a regular ring.

If Z is irr of codim 1 and η_Z be the generic point of Z (i.e. $\overline{\{\eta_Z\}} = Z$), $\dim \mathcal{O}_{Z,\eta_Z} = 1$.

Indeed, recall for $x \in X$, $\text{codim}_X(\overline{\{x\}}) = \dim \mathcal{O}_{X,x}$.

The codim of a point $x \in X$ is by definition $\dim \mathcal{O}_{X,x} = \text{codim}_X(\overline{\{x\}})$.

§: Divisor of a rational function.

Recall: $K(X) =$ Field of rational function of X

$= \mathcal{O}_{X,\eta}$ where η is the generic pt of X .

$$= \left\{ \frac{f}{g} \in \mathcal{O}_X(U) \mid U \subseteq X \right\} / \frac{f}{g} \in \mathcal{O}_X(U) \sim \frac{f}{g} \in \mathcal{O}_X(V) \text{ iff } \exists V \subseteq U \cap V \text{ s.t. } \frac{f}{g}|_V = \frac{f}{g}|_V$$

Given an $Z \subseteq X$, $\text{codim } 1, \text{ irr}$ closed

The local ring $\mathcal{O}_{X,\eta_Z}$ (η_Z gen pt of Z) is reg of dim 1. The maximal ideal is gen by 1 elt. Choose a gen $\mathfrak{f}$. For

is sub of dim 1, The maximal ideal is
 gen by 1 elt. Choose a gen f . For
 $d \in \mathcal{O}_{X, \eta}$, define

$$\text{ord}_Z(d) = \max \{n \mid d \in (f^n)\}$$

note, $\text{ord}_Z(d)$ does not depend on the choice of
 The gen f , as any other gen is $f \cdot u$ where
 u is a unit.

Since $K(X) = \text{Frac}(\mathcal{O}_{X, \eta_Z})$, extend ord_Z
 to $K(X) \rightarrow \mathbb{Z}$ by setting $\text{ord}_Z(d/p)$
 $= \text{ord}_Z(d) - \text{ord}_Z(p)$

for $d, p \in \mathcal{O}_{X, \eta_Z}$ and $\text{ord}_Z(0) = \infty$.

Prop / Def. For $f \in K(X)^*$,

$$\sum_{\substack{Z \text{ irr of} \\ \text{codim } 1}} \text{ord}_Z(f) \cdot Z \text{ is an Weil divisor.}$$

This is called The divisor of f and denoted $\text{div}(f)$.

Pf. Since X is noeth, choose a finite affine
 open cover $X = \bigcup_{i=1}^n U_i$.

For each i , write $f = f_i/g_i$. For an irr closed
 subset Z of codim 1 such that $Z \cap U_i \neq \emptyset$ or
 equivalently s.t. The gen f of $Z, \eta_Z \in U_i$,
 $\text{ord}_Z(f) \neq 0 \Rightarrow$ either $\text{ord}_Z(f_i) \neq 0$ or $\text{ord}_Z(g_i) \neq 0$.

Now for $h \in \mathcal{O}_X(U_i)$, and a codim 1 pt
 $p \in \text{Spec}(\mathcal{O}_X(U_i)) = U_i$, $\text{ord}_p(h) \neq 0$
 $\Leftrightarrow h$ is not a unit in
 $\mathcal{O}_X(U_i)_p$
 $\Leftrightarrow h \in p \mathcal{O}_X(U_i)_p$
 $\Leftrightarrow h \in p$ in $\mathcal{O}_X(U_i)$

Now by Krull's thm The set
 $\{q \in \text{Spec}(\mathcal{O}_X(U_i)) \mid h \in q\}$ is finite since
 each such prime is a min prime of $\mathcal{O}_X(U_i)/h$.

So $\#$ of codim 1 pts in $\mathcal{O}_X(U_i)$ s.t.
 $\text{ord}_{\overline{\zeta}}(f_i) \neq 0$ or $\text{ord}_{\overline{\zeta}}(g_i) \neq 0$ is finite.

For an irr closed $Z \subseteq X$, $\text{ord}_Z(f) \neq 0$
 $\Rightarrow \exists i$ $\text{ord}_{Z \cap U_i}(f_i/g_i) \neq 0$ for some i
 $\Rightarrow \sum \text{ord}_Z(f) \cdot Z$ is a finite sum

Remk. For a codim 1 irr subset Z and $f \in K(X)$,
 $\text{ord}_Z(f)$ measures the order of zeros and poles
 f along Z . Indeed since $K(X) = \text{Frac}(\mathcal{O}_{X, \eta_Z})$
 and $\mathfrak{m}_Z$ is a principal ideal (π) ,
 $\text{ord}_Z(f) = n$ where $u \in \mathcal{O}_{X, \eta_Z}^*$

$\nexists$ along Z .
and $\mathfrak{m}_Z$ is a principal ideal $(\cdot t)$.
any $f = t \cdot \text{ord}_Z(f) \cdot u$ where $u \in \mathcal{O}_{X, \eta_Z}^*$

Prop: For any irr closed Z of codim 1,
 $\text{ord}_Z(fg) = \text{ord}_Z(f) + \text{ord}_Z(g)$.

- Thus $\text{div}(fg) = \text{div}(f) + \text{div}(g)$.
- $\text{div}(1) = 0$ • $\text{div}(1/f) = -\text{div}(f)$
- Thus $\{\text{div}(f) \mid f \in K(X)^*\} \subseteq \text{Weil}(X)$
is a subgp.

Def: The quotient gp $\frac{\text{Weil}(X)}{\{\text{div}(f) \mid f \in K(X)^*\}}$ is
called the Weil divisor class gp or simply the
divisor class gp and denoted $\text{cl}(X)$.

Ex-1) $\text{cl}(A_k^n) = \{0\}$ where k is a field.
 $A_k^n = \text{Spec}(k[x_1, \dots, x_n])$. Z irr closed
of codim 1 $\Rightarrow \exists$ an irr fct f s.t
 $Z = V(f)$. Note $\text{div}(f) = Z$.
So $K(A_k^n) \xrightarrow{\text{div}} \text{Weil}(A_k^n)$ is sur.

Ex-2: $\text{cl}(\mathbb{P}_k^n) \cong \mathbb{Z}$. Given a codim 1 irr closed
 Z , $\exists$ a homo irr fct f , such that
 $Z = V(f)$, define $\deg Z = \deg f$.

Have a map. $\deg: \text{Weil}(\mathbb{P}_k^n) \rightarrow \mathbb{Z}$. ($\deg(V(x_0)) = 1$)

Given $D \in \text{Weil}(\mathbb{P}_k^n)$, write

$$D = \sum_{i=1}^r n_i V(f_i) - \sum_{j=1}^m m_j V(g_j) \text{ where}$$

$$n_i, m_j > 0.$$

$$\deg D = \sum_{i=1}^r n_i \deg(f_i) - \sum_{j=1}^m m_j \deg(g_j) = 0$$

$$\Leftrightarrow \deg\left(\prod_{i=1}^r f_i^{n_i}\right) = \deg\left(\prod_{j=1}^m g_j^{m_j}\right)$$

$$\Leftrightarrow R = \frac{\prod_{i=1}^r f_i^{n_i}}{\prod_{j=1}^m g_j^{m_j}} \in K(\mathbb{P}_k^n) \text{ and}$$

$$D = \text{div}(R).$$

So $\deg$ induces an isom $\text{cl}(\mathbb{P}_k^n) \cong \mathbb{Z}$.
[$V(x_0)$ being a gen.]

End of 15.11.24 lecture

Def. Y integral scheme. $K(Y)$ is the constant
sheaf $K(Y)(U) = K(Y)$.

Def. Given $D \in \text{Weil}(X)$, define the sheaf $\mathcal{O}_X(D) \subseteq K(X)$
s.t. $\{f \in K(X) \mid \text{div}(f) + D \geq 0\} \subseteq K(X)(U) = K(X)$

sheaf $\mathcal{O}_X(D)$

Def. Given $D \in \text{Div}(X)$, define the sheaf $\mathcal{O}_X(D) \subseteq \underline{K}(X)$ by $\mathcal{O}_X(D)(U) = \{f \in K(X)^* \mid (\text{div}(f) + D)|_U \geq 0\} \subseteq K(X)(U) = K(X)$

where for $D' = \sum n_i Z_i \in \text{Div}(X)$,

$\mathcal{O}_X(D') = \sum n_i (Z_i \cap U) \in \text{Div}(U)$; $\bullet D' \geq 0 \iff n_i \geq 0 \forall i$.

Proof. $\mathcal{O}_X(D) \subseteq \underline{K}(X)$ is an additive subgp sheaf

pf. It is straight forward to check $\mathcal{O}_X(D)$ is a sheaf.

We verify $\mathcal{O}_X(D)(U) \subseteq K(X)$ is a subgp.
Let $f_1, f_2 \in \mathcal{O}_X(D)(U)$ and $Z \subseteq \text{codim} \geq 1$ such that $Z \cap U \neq \emptyset$. Choose a generator of the maximal ideal $\mathcal{O}_{X, \eta_Z}$, denoted t .

$t^{\text{ord}_Z(f_1)} \mid f_1, t^{\text{ord}_Z(f_2)} \mid f_2$ in $\mathcal{O}_{X, \eta_Z}$.

Thus $t^{\min\{\text{ord}_Z(f_1), \text{ord}_Z(f_2)\}} \mid f_1 + f_2$

$\Rightarrow \text{ord}_Z(f_1 + f_2) \geq \min\{\text{ord}_Z(f_1), \text{ord}_Z(f_2)\}$.

$\Rightarrow$ If $(D + \text{div}(f_1))|_U \geq 0, (D + \text{div}(f_2))|_U \geq 0$. Then $(D + \text{div}(f_1 + f_2))|_U \geq 0 \Rightarrow f_1 + f_2 \in \mathcal{O}_X(D)(U)$.

Since $\text{ord}_Z(-f) = -\text{ord}_Z(f)$, $-f \in \mathcal{O}_X(D)(U)$.

Thus $(\mathcal{O}_X(D)(U), +)$ is a group.

Rem. Say $D = \sum_{i=1}^r n_i Z_i$, $\text{div } f = \sum_{i=1}^r m_i Z_i$, $m_i = \text{ord}_{Z_i}(f)$.

$f \in \mathcal{O}_X(D)(U) \iff$ for each i s.t. $Z_i \cap U \neq \emptyset$, $n_i + \text{ord}_{Z_i}(f) \geq 0$

$\iff$ if $n_i \geq 0$, f can have a pole of order at most n_i along Z_i , if $n_i \leq 0$, $\text{ord}_{Z_i}(f) \geq -n_i$ i.e. f must have a zero of ord at least n_i along Z_i .

Cartier divisors: Y be an integral scheme

Def. Y integral $\underline{K}(Y)^*$ is the constant sheaf of abelian gps $\underline{K}(Y)^*(U) = K(Y)^*$, under plt.

$\bullet \mathcal{O}_Y^*$ is the sub sheaf of abelian gps given by $\mathcal{O}_Y^*(U) = \text{units of the ring } \mathcal{O}_Y(U)$.

2.1 The group (abelian) of Cartier divisors is

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Def. The group (abelian) of Cartier divisors is

The group $\Gamma(Y, \underline{K(Y)^*} / \mathcal{O}_Y^*) = \text{Cart}(Y)$

• An elt of $\Gamma(Y, \underline{K(Y)^*} / \mathcal{O}_Y^*)$ is called a Cartier divisor.

• There is a natural map from

$$K(Y)^* \longrightarrow \text{Cart}(Y)$$

The quotient gr, denoted $\text{CaCl}(Y)$, is called the Cartier divisor class group

Prop. Given $s \in \text{Cart}(Y)$, $\exists$ an open covering $Y = \bigcup_{i=1}^r U_i$ and $f_i \in K(Y)^*$ s.t. $s|_{U_i}$ is represented by f_i . On $U_i \cap U_j$, $f_i = u_{ij} f_j$ for some $u_{ij} \in \mathcal{O}_Y^*(U_i \cap U_j)$

$\times X$ noether, integral, reg in codim 1.

Def/Prop. Given Z irred local of codim 1,

$$(\underline{K(X)^*} / \mathcal{O}_X^*)_{\eta_Z} \cong \frac{K(X)^*_{\eta_Z}}{(\mathcal{O}_X^*)_{\eta_Z}} = \frac{K(X)^*}{\mathcal{O}_{X, \eta_Z}^*}$$

For $R \in \mathcal{O}_{X, \eta_Z}^*$, $\text{ord}_Z(R) = 0$.

Thus $\text{ord}_Z: K(X)^* \rightarrow \mathbb{Z}$ factors through

$$K(X)^* / \mathcal{O}_{X, \eta_Z}^*$$

• Given $s \in \text{Cart}(X)$, define $\text{ord}_Z(s) = \text{ord}_Z(s_{\eta_Z})$ where $s_{\eta_Z} \in \frac{K(X)^*}{\mathcal{O}_{X, \eta_Z}^*}$

• Given $s \in \text{Cart}(X)$, define

$$\text{div}(s) = \sum \text{ord}_Z(s) \cdot Z \in \text{Weil}(X).$$

• The following claim gives a way to 'think about' $\text{div}(s)$ for $s \in \text{Cart}(X)$ and justifies why the sum on the right hand is finite.

Claim. Given $s \in \text{Cart}(X)$, choose a finite open covering $X = \bigcup_{i=1}^r U_i$ such that for each i , $\exists f_i \in K(X)^*$ satisfying $s|_{U_i} = f_i$ in $\underline{K(X)^*} / \mathcal{O}_X^*(U_i)$

Then $\text{div}(s)|_{U_i} = \text{div}(f_i)|_{U_i}$

$\Rightarrow \text{div}(s) = \sum \text{div}(f_i)$ such that $Z \cap U_i \neq \emptyset$.

Then $\text{div}(f)|_{U_i} = \text{div}(f_i)$

Pf. For $Z \subseteq X$ such that $Z \cap U_i \neq \emptyset$.
 $\text{ord}_Z(f) = \text{ord}_Z(f_i)$. So the claim follows.

Prop. (i) $\text{div}: \text{Cart}(X) \rightarrow \text{Weil}(X)$ is a gr hom.

(ii) For $f \in K(X)^*$, $\text{div}(f \in \text{Cart}(X)) = \text{div}(f(K(X)^*))$

So div induces a gr hom.

$$\text{div}: \text{Cl}(X) := \frac{\text{Cart}(X)}{K(X)^*} \longrightarrow \text{Cl}(X)$$

(iii) When X is normal, div and div are injective.

Def. A Weil divisor D on X is called locally principal if $\exists$ a (finite) open covering $X = \bigcup U_i$ and $f_i \in K(X)^*$ such that $D|_{U_i} = \text{div}(f_i|_{U_i})$ for each i .

A Weil divisor D is called effective if $D = \sum_{i=1}^n n_i Z_i$ where $n_i \geq 0$

Prop. 1) $\text{div}(\text{Cart}(X)) \subseteq \text{locally principal Weil divisors}$.

2) When X is normal, the above containment is an equality.

Pf. 1) follows from claim *.

2) Given a locally principal Weil divisor D , choose a finite open covering $X = \bigcup_{i=1}^n U_i$ and $f_i \in K(X)^*$ such that $D|_{U_i} = \text{div}(f_i|_{U_i})$. Since $\text{div}(f_i|_{U_i \cap U_j}) = \text{div}(f_j|_{U_i \cap U_j})$,
 $\text{div}(f_i/f_j)|_{U_i \cap U_j} = 0$. Since X is normal,
 $f_i/f_j \in \mathcal{O}_X(U_i \cap U_j)^*$. So $\{f_i \in K(X)^* / \mathcal{O}_X(U_i)^*\}_i$
 glue to give a global section

$\tau_i / f_i \in \mathcal{O}_X(U_i) \rightarrow \dots \rightarrow \mathcal{O}_X$
 glue to give a global section
 $s \in \Gamma(X, \underline{K(X)} / \mathcal{O}_X)$.

By claim $\star$, $\text{div}(s) = D$.

- We examine surjectivity of $\text{Cart}(X) \xrightarrow{\text{div}} \text{Weil}(X)$.
 The following Thm is crucial.

Thm. Let A be a noetherian ring.
 A is a unique factorization domain
 $\iff A$ is normal and $\mathcal{C}(\text{Spec } A) = \{0\}$.

Pf. See Prop 6.2, Hart.

Thm. Let X be a normal, noetherian scheme
 $\text{Cart}(X) \xrightarrow{\text{div}} \text{Weil}(X)$ is surjective (eq. bijective)
 $\iff X$ is locally factorial, i.e. for every affine $U \subseteq X$
 $\mathcal{O}_X(U)$ is a U.F.D

$\implies \overline{\text{div}} : \text{CaCl}(X) \xrightarrow{\sim} \mathcal{C}(X)$.

Pf. We only verify $\Leftarrow$ of the tot $\iff$.
 We know $\text{div}(\text{Cart}(X)) = \text{locally principal Weil divisors of } X$.

Given $D \in \text{Weil}(X)$ and $U \subseteq X$ affine open,
 $D|_U \in \text{Weil}(U)$. But the Thm above $\mathcal{C}(U) = \{0\}$,
 so $\exists f \in K(U)^\times = K(X)^\times$ such that $D|_U = \text{div } f|_U$.
 So D is locally principal.

Thm. When X is regular, $\text{div} : \text{Cart}(X) \xrightarrow{\sim} \text{Weil}(X)$
 and $\overline{\text{div}} : \text{CaCl}(X) \xrightarrow{\sim} \mathcal{C}(X)$.

$\S$ Picard group and CaCl

The results of this section are true for any integral scheme, which are not necessarily locally

integral scheme, which are not necessarily every
math.

Thm. Y be an integral scheme. Given a

$s \in \text{Cart}(Y)$, define

$$\mathcal{L}(s)(U) = \left\{ \frac{f}{s_x} \in K(Y)^* \mid \frac{f}{s_x} \in \mathcal{O}_x(U) - \{0\} / \mathcal{O}_{Y,x}^* \right\} \cup \{0\}$$

$$\forall x \in U$$

$$\subseteq \underline{K(Y)}(U) = K(Y)$$

Then 1) $\mathcal{L}(s)$ is a sub $\mathcal{O}_Y$ -mod sheaf of the
sheaf of abelian grps $(\underline{K(Y)}, +)$.

2) $\mathcal{L}(s)$ is invertible

Pf 1) is straight forward.

2) Claim Given $s \in \text{Cart}(Y)$. Choose a covering

$$Y = \bigcup_{i \in I} U_i \text{ s.t. } s|_{U_i} = f_i$$

$$\uparrow_{i \in \Gamma(U_i, K(Y)^* / \mathcal{O}_Y^*)}$$

$$\text{Then } \mathcal{L}(s)|_{U_i} \xleftarrow{\sim} \mathcal{O}_{U_i}.$$

$$\downarrow \frac{1}{f_i} \longleftarrow 1$$

$$\text{In particular } \mathcal{L}(s)(U_i) = \frac{1}{f_i} \mathcal{O}_Y(U_i) \text{ and}$$

$$\mathcal{L}(s)|_{U_i} \xrightarrow{\sim} \frac{1}{f_i} \mathcal{O}_Y(U_i) \text{ when } U_i \text{ is affine.}$$

Pf $\frac{1}{f_i} \in \Gamma(U_i, \mathcal{L}(s))$ as $f_i \cdot \frac{1}{f_i} \in \mathcal{O}_{Y,y}^*$
 $\forall y \in U_i$

$$\begin{array}{ccc} \mathcal{L}(s)|_{U_i} & \xrightarrow{\cdot f_i} & \mathcal{O}_{U_i} \\ \cup & & \cup \\ \underline{K(Y)} & \xrightarrow{f_i} & \underline{K(Y)} \end{array}$$

gives the inverse.

Thm. Let Y be an integral scheme.

$\mathcal{L}(s)$ gives a bijection

Thm. Let Y be an integral scheme.

1) The map $s \mapsto \mathcal{L}(s)$ gives a bijection
 $\text{Cart}(Y) \longleftrightarrow \{ \text{invertible subsheaves of } \underline{K(Y)} \}.$

2) The map in (1) induces an isom of groups
 $\text{Coc}(Y) \xrightarrow{\sim} \{ \text{isom classes of invertible subsheaves of } \underline{K(Y)} \}.$

where the group structure on the right again comes from $\otimes_{\mathcal{O}_Y}$

3) The map induced by (2)
 $\text{Coc}(Y) \xrightarrow{(2)} \{ \text{isom classes of invertible subsheaves of } \underline{K(Y)} \}$

$\swarrow$
 $\text{Pic}(Y)$

is an isom. of groups.

Pf. 1) Injection: Assume $s_\alpha, s_\beta \in \text{Cart}(Y)$ have the same image. Choose an open covering

$$Y = \bigcup_{i \in I} U_i$$

such that $\exists \varphi_i^\alpha, \varphi_i^\beta$ for each i

satisfying, $s_\alpha|_{U_i} = \varphi_i^\alpha$, $s_\beta|_{U_i} = \varphi_i^\beta \quad \forall i.$

$$\text{Thus } \frac{1}{\varphi_i^\alpha} \mathcal{O}_Y(U_i) = \frac{1}{\varphi_i^\beta} \mathcal{O}_Y(U_i) \quad \forall i$$

$$\Rightarrow \varphi_i^\beta / \varphi_i^\alpha, \varphi_i^\alpha / \varphi_i^\beta \in \mathcal{O}_Y(U_i) \quad \forall i$$

$$\Rightarrow \varphi_i^\beta / \varphi_i^\alpha \in \mathcal{O}_Y(U_i)^\times \quad \forall i$$

$$\Rightarrow s_\alpha = s_\beta \in \Gamma(Y, \underline{K(Y)}^\times / \mathcal{O}_Y^\times).$$

2) Surjection: Given an invertible $\mathcal{O}_Y$ subsheaf $\mathcal{F}$ of

$\rightarrow$ surjectivity: Given an invertible $\mathcal{O}_Y$ submod $\mathcal{F}$ of $\underline{K(Y)}$, choose an open covering $Y = \bigcup_{i \in I} U_i$ such that $\mathcal{F}|_{U_i} = f_i \cdot \mathcal{O}_{U_i}$ for some $f_i \in \underline{K(Y)}^\times$.

Since $f_i \cdot \mathcal{O}_{U_i}|_{U_i \cap U_j} = f_j \cdot \mathcal{O}_{U_j}|_{U_i \cap U_j}$

$\Rightarrow \left\{ 1/f_i \in \Gamma(U_i, \underline{K(Y)}^\times / \mathcal{O}_Y^\times) \right\}_{i \in I}$ glue to give a section $s \in \Gamma(Y, \underline{K(Y)}^\times / \mathcal{O}_Y^\times)$

By claim **, $\mathcal{L}(s) = \mathcal{F}$.

2) By claim **, we have $\mathcal{L}(s_1) \otimes_{\mathcal{O}_Y} \mathcal{L}(s_2) \xrightarrow{\sim} \mathcal{L}(s_1 s_2)$

So we only need to check: if $\mathcal{L}(s) \xrightarrow{\sim} \mathcal{O}_Y$, then $\exists f \in \underline{K(Y)}^\times$ such that $s = f$ in $\Gamma(Y, \underline{K(Y)}^\times / \mathcal{O}_Y^\times)$.

To that end, fix an isom $\mathcal{O}_Y \xrightarrow{\sim} \mathcal{L}(s)$.

Denote the image of $1 \in \mathcal{O}_Y(Y)$ via this isom by $g \in \underline{K(Y)}^\times$.

So $\mathcal{L}(s)(u) = g \cdot \mathcal{O}_Y(u) \quad \forall u \subseteq_{\text{open}} Y$.

Choose an open cover $Y = \bigcup_{i \in I} U_i$ s.t. $s|_{U_i} = f_i$ for some $f_i \in \underline{K(Y)}^\times$.

By claim **, $\mathcal{L}(s)(U_i) = 1/f_i \cdot \mathcal{O}_Y(U_i)$

Then $1/f_i \cdot \mathcal{O}_Y(U_i) = g \cdot \mathcal{O}_Y(U_i) \quad \forall i$ $1/f_i = g \cdot y$
 $\Rightarrow g / 1/f_i \in \mathcal{O}_Y(U_i)^\times \quad \forall i$ $g = 1/f_i \cdot y$

$\Rightarrow s = 1/g \in \Gamma(Y, \underline{K(Y)}^\times / \mathcal{O}_Y^\times) \quad \square$

$\therefore$ therefore once we show every invertible mod is...

3) follows once we show every invertible $\mathcal{O}_Y$ -mod is isom to some invertible $\mathcal{O}_Y$ -mod of $\underline{K(Y)}$. Given an invertible $\mathcal{O}_Y$ -mod $\mathcal{L}$, we produce an injective $\mathcal{O}_Y$ -mod map $\mathcal{L} \rightarrow \underline{K(Y)}$. The image will be the desired invertible $\mathcal{O}_Y$ -submod isom to $\mathcal{L}$.

To that end, choose an isom $\mathcal{L}_\eta \xrightarrow[\cong]{\sim} \underline{K(Y)}$ where η is the gen pt of Y .

For $U \subseteq_{\text{open}} Y$, the canonical map composed with η

$$\mathcal{L}(U) \rightarrow \mathcal{L}_\eta \xrightarrow[\cong]{\sim} \underline{K(Y)}$$

gives a map of

$$\Psi_U: \mathcal{L}(U) \rightarrow \mathcal{L}_\eta \xrightarrow[\cong]{\sim} \underline{K(Y)}(U) = \underline{K(Y)}$$

Ψ_U induces a $\mathcal{O}_Y$ -lin map $\mathcal{L} \rightarrow \underline{K(Y)}$.

Since Y is integral, $\mathcal{L}(U) \hookrightarrow \mathcal{L}_\eta$ and hence Ψ_U is inj $\forall U$. $\square$.